RHEOLOGICAL BEHAVIOR OF A DILUTE SUSPENSION OF SPHERICAL PARTICLES IN A NON-NEWTONIAN LIQUID

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To establish the rheological behavior of the media under consideration we investigate the perturbations introduced into the flow of a Reiner–Rivlin liquid with a parallel velocity gradient (uniaxial stretching) by a spherical drop of a Newtonian liquid. We obtain the following boundary-value problem:

$$T_{ij,j} = 0, \quad v_{i,i} = 0; \tag{1}$$

$$T_{ij} = -p\delta_{ij} + \mu_1 E_{ij} + \mu_3 E_{ik}E_{kj}; \tag{2}$$

$$v_x = -\frac{q}{2}x, \quad v_y = -\frac{q}{2}y, \quad v_z = qz, \quad p = p_\infty \quad \text{as} \quad r \to \infty; \tag{3}$$

$$v_r = 0, \quad v_\theta = v_\theta^*, \quad T_{r\theta} = T_{r\theta}^* \quad \text{at} \quad r = a. \tag{4}$$

Relations (1) are the dynamical equations in the stresses in the Stokes approximation and the equation of continuity; relation (2) is the rheological equation of state of a dispersion medium; Eqs. (3) are the boundary conditions at infinity (unperturbed flow — uniaxial stretching); and Eqs. (4) are the boundary conditions at the surface of a particle (impenetrability of the surface, continuity of the tangential component of the velocity and $T_{r\theta}$ at this surface). Here T_{ij} is the stress tensor, v_i is the velocity, p is the pressure, δ_{ij} is the Kronecker symbol, E_{ij} is twice the rate of strain tensor, μ_1 and μ_3 are, respectively, the viscosity and cross viscosity of the dispersion medium, q is the rate of stretching, p_∞ is the pressure in the unperturbed flow, x, y, z and r, θ, φ are, respectively, Cartesian and spherical coordinate systems with the origin at the center of the particle, a is the radius of the particle, and an asterisk indicates that the quantity is taken from the solution of the problem of motion of the liquid inside the particle, i.e., simultaneously with the boundary-value problem (1)-(4), it is necessary to solve the problem of the motion of a Newtonian liquid inside the particle produced by the flow of the dispersion medium under study and to match the solutions at the surface of separation of the internal and external problems.

Suppose μ_1 and μ_3 are constants and the dimensionless parameter $\varepsilon = (\mu_3/\mu_1)q \ll 1$. This restricts the class of dispersion systems considered, but makes it possible to linearize the boundary-value problem (1)-(4). The steady flow of a Reiner–Rivlin liquid past a sphere under the above assumptions was studied in [1, 2]. Kato et al. [3] cite a number of polymer solutions which satisfy the equation of state (2) for contant μ_1 and μ_3.

Introducing the stream function ψ related to v_r and v_θ by the equations

$$v_r = -\frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}, \quad v_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r},$$

and going over to dimensionless quantities with the scales $r \sim a$, $v_i \sim aq$, $E_{ij} \sim q$, $T_{ij} \sim \mu_1 q$, $p \sim p_\infty$, and $\psi \sim a^3 q$, we seek the solution of the boundary-value problem (1)-(4) in the form of asymptotic expansions in powers of a small parameter,

$$\psi = \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \ldots,$$
$$p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \ldots. \tag{5}$$

The symbol indicating the dimensionless character of the quantities in (5) is omitted from now on.

In the zero approximation we obtain a boundary-value problem corresponding to the flow (3) of a Newtonian liquid with a dynamic viscosity μ_1 past a spherical drop. This problem has the solution

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$$\psi_0 = (-r^3/2 + A + B/r^4)\sin^2\theta\cdot\cos\theta; \tag{6}$$

$$p_0 = 1 - (2A/r^3)(2 - 3\sin^2\theta), \tag{7}$$

where $A = (2 + 5\sigma)/4(1 + \sigma)$; $B = -3\sigma/4(1 + \sigma)$; $\sigma = \mu_*/\mu_1$; μ_* is the dynamic viscosity of the material of the particle.

In the first approximation for the external problem we obtain the equations

$$\begin{aligned}\frac{\partial p}{\partial r} &= \frac{\partial E_{rr}^{(1)}}{\partial r} + \frac{1}{r}\frac{\partial E_{r\theta}^{(1)}}{\partial\theta} + \frac{1}{r}\left(2E_{rr}^{(1)} - E_{\theta\theta}^{(1)} - E_{\varphi\varphi}^{(1)} + E_{r\theta}^{(1)}\operatorname{ctg}\theta\right) + \\ &+ \frac{\partial}{\partial r}\left(E_{rr}^{(0)2} + E_{r\theta}^{(0)2}\right) - \frac{1}{r}\frac{\partial}{\partial\theta}\left(E_{r\theta}^{(0)}E_{\varphi\varphi}^{(0)}\right) + \frac{1}{r}\left(2E_{rr}^{(0)2} + E_{r\theta}^{(0)2} - E_{\theta\theta}^{(0)2} - E_{\varphi\varphi}^{(0)2} - E_{r\theta}^{(0)}E_{\varphi\varphi}^{(0)}\operatorname{ctg}\theta\right), \\ \frac{\partial p_1}{\partial\theta} &= r\frac{\partial E_{r\theta}^{(1)}}{\partial r} + \frac{\partial E_{\theta\theta}^{(1)}}{\partial\theta} + 3E_{r\theta}^{(1)} + \left(E_{\theta}^{(1)} - E_{\varphi\varphi}^{(1)}\right)\operatorname{ctg}\theta - r\frac{\partial}{\partial r}\left(E_{r\theta}^{(0)}E_{\varphi\varphi}^{(0)}\right) + \\ &+ \frac{\partial}{\partial\theta}\left(E_{r\theta}^{(0)2} + E_{\theta\theta}^{(0)2}\right) - 3E_{r\theta}^{(0)}E_{\varphi\varphi}^{(0)} + \left(E_{r\theta}^{(0)2} + E_{\theta\theta}^{(0)2} - E_{\varphi\varphi}^{(0)2}\right)\operatorname{ctg}\theta.\end{aligned} \tag{8}$$

Eliminating p_1 in (8) and substituting $E_{ij}^{(0)}$ calculated from (6), we obtain the following adjoint problem:

$$\left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2\psi_1 = \left(\frac{576A^2}{r^7} + \frac{1920AB}{r^9}\right)\sin^2\theta\cos\theta - \left(\frac{720A^2}{r^7} + \frac{2160AB}{r^9}\right)\sin^4\theta\cos\theta; \tag{9}$$

$$\left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2\psi_1^* = 0; \tag{10}$$

$$\psi_1 < 0\,(r^2),\quad \frac{\partial\psi_1}{\partial r} < 0\,(r)\quad \text{as}\quad r\to\infty,\quad \psi_1 = 0,\quad \psi_1^* = 0,\quad \frac{\partial\psi_1}{\partial r} = \frac{\partial\psi_1^*}{\partial r}; \tag{11}$$

$$\sigma r\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial\psi_1^*}{\partial r}\right) = r\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial\psi_1}{\partial r}\right) - E_{r\theta}^{(0)}E_{\varphi\varphi}^{(0)}\sin\theta\quad \text{at}\quad r = 1. \tag{12}$$

We seek the solution of problem (9)-(12) in the form

$$\psi_1 = f_1(r)\sin^2\theta\cos\theta + f_2(r)\sin^4\theta\cos\theta,$$

$$\psi_1^* = f_1^*(r)\sin^2\theta\cos\theta + f_2^*(r)\sin^4\theta\cos\theta.$$

We obtain the following system of ordinary differential equations for $f_1(r)$, $f_2(r)$, $f_1^*(r)$, and $f_2^*(r)$:

$$\left.\begin{aligned} &f_1^{IV} - \frac{12}{r^2}f_1'' + \frac{24}{r^3}f_1' + \frac{16}{r^2}f_2'' - \frac{32}{r^3}f_2' - \frac{160}{r^4}f_2 = \frac{576A^2}{r^7} + \frac{1920AB}{r^9}, \\ &f_2^{IV} - \frac{40}{r^2}f_2'' + \frac{80}{r^3}f_2' + \frac{280}{r^4}f_2 = -\frac{720A^2}{r^7} - \frac{2160AB}{r^9};\end{aligned}\right\} \tag{13}$$

$$\left.\begin{aligned} &f_1^{*IV} - \frac{12}{r^2}f_1^{*''} + \frac{24}{r^3}f_1^{*'} + \frac{16}{r^2}f_2^{*''} - \frac{32}{r^3}f_2^{*'} - \frac{160}{r^4}f_2^* = 0, \\ &f_2^{*IV} - \frac{40}{r^2}f_2^{*''} + \frac{80}{r^3}f_2^{*'} + \frac{280}{r^4}f_2^* = 0.\end{aligned}\right\} \tag{14}$$

Solving Eqs. (13) and (14) with boundary conditions rewritten for $f_1(r)$, $f_2(r)$, $f_1^*(r)$, and $f_2^*(r)$, from (11) and (12) we find

$$\psi_1 = \left[C + \frac{1}{r^2}C_1 - \frac{4A^2}{r^3} + \frac{4AB}{r^5}\right]\sin^2\theta\cdot\cos\theta + \left[\frac{1}{r^2}C_2 + \frac{9A^2}{r^3} + \frac{1}{r^4}C_3 - \frac{6AB}{r^5}\right]\sin^4\theta\cdot\cos\theta, \tag{15}$$

where

$$C = \frac{-10A\sigma(A - 3B) - 16A^2 + 60AB + 15B + 20B^2}{5(1+\sigma)};$$

the C_i are known functions of σ.

Solving Eqs. (8) with the boundary condition $p_1 = 0$ as $r\to\infty$, we obtain

$$p_1 = \frac{4(5A - C) - 6(17A - C)\sin^2\theta + 90A\sin^4\theta}{r^3}, \tag{16}$$

where the terms retained ensure the determination of the rheological characteristics of the suspension with an accuracy to quantities of the order of the volume concentration of the suspended particles.

By using Eqs. (6), (7), (15), and (16), we determine the dissipation of mechanical energy in a volume of the dispersion medium bounded by the surface of a particle and a spherical surface σ_0 of radius R >> r in terms of the forces applied to this surface,

$$W = \int_{\sigma_0} (P_{rr}v_r + P_{r\theta}v_\theta)\, d\sigma. \tag{17}$$

Here and below all quantities are written in dimensionless form.

The dissipation of mechanical energy in a selected element of the medium divided by its volume τ, determined with an accuracy to quantities of the order of the volume concentration of the suspended particles, has the form

$$W = \frac{w}{\tau} = 3\mu_1 q^2\left(1 + \frac{2}{5}A\Phi\right) + 3\mu_3 q^3\left[1 - \frac{2}{5}(A - C)\,\Phi\right], \tag{18}$$

where Φ is the volume concentration of the suspended particles.

Since the suspension under consideration is dilute, the suspended particles are spherical, and for $\varepsilon \ll 1$ the properties of the dispersion medium are little different from those of a Newtonian liquid (or the rate of strain is small), we assume that the rheological equation of state of the suspension has the form

$$T_{ij} = -p\delta_{ij} + \mu_{1ef}E_{ij} + \mu_{3ef}E_{ik}E_{kj}, \tag{19}$$

where μ_{1ef} and μ_{3ef}, the viscosity and cross viscosity of the suspension, are to be determined.

Then W can be found, in terms of the internal forces in the following way:

$$W = \frac{1}{2}\overline{T}_{ij}\overline{E}_{ij},$$

where $\overline{T}_{ij}$ and $\overline{E}_{ij}$ are, respectively, the stress tensor (19) and twice the rate of strain tensor averaged over a selected volume element of the medium under study.

The components of the tensor $\overline{E}_{ij}$, determined with the same accuracy as assumed in (18), are

$$\overline{E}_{xx} = \overline{E}_{yy} = \frac{2}{\tau}\int_\tau \frac{\partial v_x}{\partial x}\, d\tau = \frac{2}{\tau}\int_\sigma v_x \frac{x}{r}\, d\sigma = -q\left(1 - \frac{4A}{5}\Phi - \frac{4C}{5}\frac{\mu_3}{\mu_1}q\Phi\right),$$

$$\overline{E}_{zz} = 2q\left(1 - \frac{4A}{5}\Phi - \frac{4C}{5}\frac{\mu_3}{\mu_1}q\Phi\right); \quad \overline{E}_{ij} = 0 \ \text{ for } \ i \neq j.$$

Hence

$$W = \frac{1}{2}\mu_{1ef}\overline{E}_{ij}\overline{E}_{ij} + \frac{1}{2}\mu_{3ef}\overline{E}_{ik}\overline{E}_{kj}\overline{E}_{ij} = 3\mu_{1ef}q^2\left(1 - \frac{8A}{5}\Phi - \frac{8C}{5}\frac{\mu_3}{\mu_1}q\Phi\right) + 3\mu_{3ef}q^3\left(1 - \frac{12A}{5}\Phi\right). \tag{20}$$

Comparing (18) and (20), we find μ_{1ef} and μ_{3ef}, and the rheological equation of state of the suspension (19) takes the form

$$T_{ij} = -p\delta_{ij} + \mu_1(1 + 2A\Phi)E_{ij} + \mu_3[1 + 2(A + C)\Phi]E_{ik}E_{kj}. \tag{21}$$

In the limit as $\sigma \to \infty$, which corresponds to a dilute suspension of rigid spherical particles, we obtain

$$T_{ij} = -p\delta_{ij} + \mu_1(1 + 2.5\Phi)E_{ij} + \mu_3(1 - 15\Phi)E_{ik}E_{kj}. \tag{22}$$

In the limit as $\sigma \to 0$, we obtain the rheological equation of state of a dilute suspension of gas bubbles,

$$T_{ij} = -p\delta_{ij} + \mu_1(1 + \Phi)E_{ij} + \mu_3(1 - 0.6\Phi)E_{ik}E_{kj}. \tag{23}$$

Thus, a dilute suspension of rigid, liquid, or gaseous spherical particles with a non-Newtonian dispersion medium which is a generalized Reiner–Rivlin liquid with constant viscosity and cross viscosity for $\varepsilon \ll 1$ (a dispersion medium slightly different from a Newtonian liquid or a small rate of strain) is itself a Reiner–Rivlin liquid with a viscosity and cross viscosity depending on the volume concentration of the suspended particles and the ratio of the viscosity of the particle material to that of the dispersion medium.

Since the situation considered in this article is very little different from the Einstein case, the coefficient 15 in Eq. (22) holds for $\Phi < 0.02$. The noticeable decrease of cross viscosity resulting from the addition of a dispersed phase to a viscoelastic liquid is well known and used in practice.

For $\mu_3 = 0$ the equations of state (21)-(23) give the classical results of the mechanics of dilute suspensions of spherical particles with a dispersion medium which is a Newtonian liquid.

It follows from the equations of state obtained that the addition of a dispersed phase with a small concentration to a Reiner–Rivlin liquid leads to a decrease in the cross viscosity, i.e., to a decrease in its non-Newtonian properties. Actually, μ_{3ef} can be written in the form

$$\mu_{3ef} = \mu_3[1 - \nu(\sigma)\Phi],$$

where $\nu(\sigma)$ lies between 0.6 and 15. The maximum value of ν corresponds to solid particles and the minimum value to gas bubbles.

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TURBULENT FLOW OF CONCENTRATED UNSTABLE EMULSIONS IN PIPES

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Unstable emulsions occur in a number of important technological processes, such as liquid extraction in oil refining and intrapipe demulsification in petroleum production. The hydrodynamic behavior of unstable emulsions differs from that of single-phase liquids in the damping of turbulent fluctuations of the dispersion medium by drops of the dispersed phase which are larger than the internal scale of turbulent fluctuations [1]. The turbulent flow of dilute unstable emulsions is described in [2].

When the content β of the dispersed phase of the emulsion lies in the range of $0.524 \leqslant \beta \leqslant 0.741$ (for $\beta = 0.741$ the phases of an unstable emulsion are inverted) the drops are closely packed, and shearing the emulsion requires an additional stress to deform them [3]:

$$\tau_0 = (0.195\beta - 0.102)\sigma/d, \quad 0.524 \leqslant \beta \leqslant 0.741$$

where σ is the interfacial tension, and d is the diameter of the drops of the emulsion. Thus, a concentrated unstable emulsion conforms to the Bingham model [4], and the equation of motion of concentrated emulsions in a pipe can be written in the form

$$\begin{gathered}(\mu_e + \mu_{ee})du/dy = \tau - \tau_0, \quad \tau_0 < \tau < \tau_w, \\ du/dy = 0, \quad \tau \leqslant \tau_0\end{gathered} \qquad (1)$$

where u and τ are, respectively, the velocity and shear stress at a distance y from the wall, τ_w is the wall shear stress, and μ_e and μ_{ee} are the dynamic and eddy viscosities of the emulsion. It is shown in [5] that the dynamic viscosity of concentrated unstable emulsions can be determined in accordance with [6] as $\mu_e = \mu_1(1 - \beta)^{-2.5}$, where μ_1 is the dynamic viscosity of the dispersion medium.

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